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Q.1 Prove that ~~a~~ group H of a group G is normal iff

$$xHx^{-1} = H, \forall x \in G$$

Proof: Let us first suppose

$$xHx^{-1} = H, \forall x \in G$$

Then, $xHx^{-1} \subseteq H, \forall x \in G$

$\Rightarrow$ H is a normal subgroup of G.

Conversely. Let H be a normal subgroup of G, then

$$xhx^{-1} \in H \;\; \forall h \in H \text{ and } \forall x \in G \qquad \text{(I)}$$

$\Rightarrow xHx^{-1} \subseteq H, \forall x \in G$

Also, let h be any element of H, then for every $x \in G$, we have

$$h = ehe = xx^{-1}hxx^{-1} = x(x^{-1}hx)x^{-1} \in xHx^{-1}$$

Since, $x^{-1}hx = x^{-1}h(x^{-1})^{-1} \in H$, $x^{-1} \in G$ and H being normal

Therefore, $H \subseteq xHx^{-1}, \forall x \in G \qquad$ (II)

Now, from I and II, we conclude that

$$xHx^{-1} = H, \forall x \in G$$

proved.

Q.2. A subgroup H of a group G is a normal subgroup of G if and only if, every left coset of H in G is a right coset of H in G.

Proof: Let us first suppose H is a normal subgroup of G. Then, we know that:

$$xHx^{-1} = H, \forall x \in G$$

$\Rightarrow (xHx^{-1})x = Hx, \forall x \in G$

$\Rightarrow (xH)x^{-1}x = Hx, \forall x \in G$

$\Rightarrow xH = Hx, \forall x \in G$

$\Rightarrow$ each left coset xH is the right coset Hx.

Conversely, let us suppose that each left coset of H in G is a right coset of H in G.

$\Rightarrow xH = Hy$, for some $x, y \in G$

Since, $e \in H$, so $x = xe \in xH$

Thus, $x \in xH = Hy$

Now, $x \in Hy \Rightarrow xy^{-1} \in H \Rightarrow Hx = Hy$

Therefore, we have

$$xH = Hx, \forall x \in G$$

$$\Rightarrow xHx^{-1} = Hxx^{-1}, \forall x \in G$$

$$\Rightarrow xHx^{-1} = H, \forall x \in G$$

Hence, H is a normal subgroup of G. proved.

Q.3. A subgroup H of a group G is a normal subgroup of G if and only if the product of two right cosets of H in G is again a right coset of H in G.

proof: Let H be a normal subgroup of a group G. and x, y be any two elements of G.

$\Rightarrow Hx, Hy$ are any two right cosets of H in G

We have, $(Hx)(Hy) = H(xH)y$ [By associativity]

$= H(Hx)y$ [$\because$ H is normal $\Rightarrow Hx = xH$]

$= (HH)xy$ [By associativity]

$= Hxy$ [$\because HH = H$]

$x, y \in G \Rightarrow xy \in G \Rightarrow Hxy$ is a right coset of H in G, thus the product of two right cosets Hx and Hy is the right coset Hxy.

Conversely: Let H be a subgroup of G such that the product of two right cosets of H in G is again a right coset of H in G.

i.e. $(Hx)(Hy) = Hxy, \forall x, y \in G$

Let $x \in G, h \in H$.

Then, $xhx^{-1} = (ex)(hx^{-1}) \in (Hx)(Hx^{-1})$

Since, $e \in H = Hxx^{-1} = He = H$, $\forall h \in H$.

Thus, we have, $xhx^{-1} \in H, \forall x \in G$, $\forall h \in H$

Therefore, H is normal subgroup of G.

Proved